# 17. Shaft Design

**Objectives**

- Compute forces acting on shafts from gears, pulleys, and sprockets.
- Find bending moments from gears, pulleys, or sprockets that are transmitting loads to or from other devices.
- Determine torque in shafts from gears, pulleys, sprockets, clutches, and couplings.
- Compare combined stresses to suitable allowable stresses, including any required stress reduction factors such as stress concentration factors and factors of safety.
- Determine suitability of shaft design and/or necessary size of shafting.



# Introduction

- Shaft must have adequate torsional strength to transmit torque and not be over stressed.
- Shafts are mounted in bearings and transmit power through devices such as gears, pulleys, cams and clutches.
- Components such as gears are mounted on shafts using keys.
- Shaft must sustain a combination of bending and torsional loads.



# Standard diameters of shafts

| Diameter (in.) | Diameter increments (in.) |
|---|---|
| Upto 3 | 1/16 |
| 3 to 5 | 1/8 |
| 5 to 8 | 1/4 |



# Torsion of circular shafts

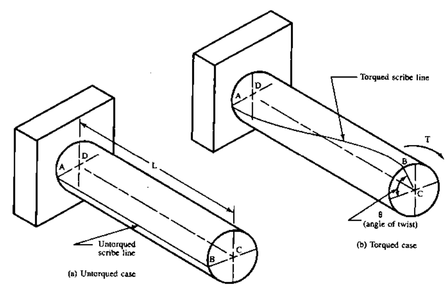



FIGURE 5.2 Untorqued versus torqued shaft configuration



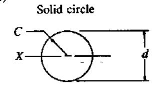
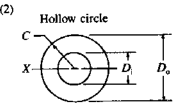
# Torsion of circular shafts

$$\text{Angle of twist, } \theta = \frac{T\,L}{G\,J}$$

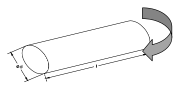
- θ = the angle of twist (radians)
- T = the applied torque (in-lb.)
- L = shaft length (in.)
- J = polar moment on inertia of the shaft cross section ($in^4$)
- G = shear modulus of elasticity of the shaft material ($lb/in^2$)



TABLE 5.2 *I* and *J* Relationships for Circular Cross-Sectional Areas

| Area Shape | *I* | *J* |
|---|---|---|
| T5.2(1) Solid circle  | $\frac{\pi d^4}{64}$ | $\frac{\pi d^4}{32}$ |
| T5.2(2) Hollow circle  | $\frac{\pi(D_o^4 - D_i^4)}{64}$ | $\frac{\pi(D_o^4 - D_i^4)}{32}$ |

## Torsional Shear Stresses

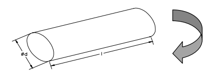


Torque

- Torsional shear stress, $S_S = \frac{T\ c}{J}$
- J = Polar moment of inertia = $\frac{\pi \times d^4}{32}$
- c = radius of the shaft
- T = Torque
- d = diameter of shaft



## Shear Stress in a shaft



Torque

- Shear stress, $S_S = \frac{16\ T}{\pi\ D^3}$

Where

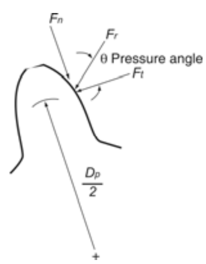
- T = torque
- D = diameter of the shaft = $\sqrt[3]{\frac{16\ T}{\pi\ S_S}}$



## Forces on spur gear teeth

- $F_t$ = Transmitted force
- $F_n$ = Normal force or separating force
- $F_r$ = Resultant force
- θ = pressure angle
- $F_n = F_t \tan\theta$

$$F_r = \frac{F_t}{\cos\theta}$$





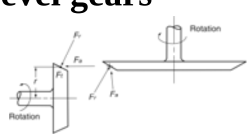

## Forces on spur gear teeth

- Power, $P = \frac{T\ n}{63{,}000}$ or $T = \frac{63{,}000\ P}{n}$
- Torque, $T = F_t\ r$ and $r = D_p/2$
- Combining the above we can write

$$F_t = \frac{2\ T}{D_p} = \frac{2\ P \times 63{,}000}{D_p\ n}$$



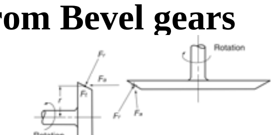
## Loads from Bevel gears




- An additional axial force will be acting on the shaft because of the bevel angle
- For the pinion it is relatively small, and can be neglected.
- For the larger gear it will be significant and will be larger than the radial separating force.

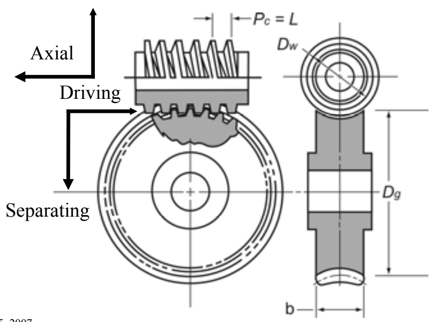


## Loads from Bevel gears




- Force transmitted, $F_n = F_t \tan\theta \cos\gamma$
- θ = Pressure angle
- γ = Cone angle
- Axial Force, $F_a = F_t \tan\theta \sin\gamma$
- Resultant Force, $F_r = \sqrt{F_t^2 + F^2}$
- $F = F_n$ or $F_a$ depending on whichever is larger

## Loads from Worm gears






## Loads from Worm gears

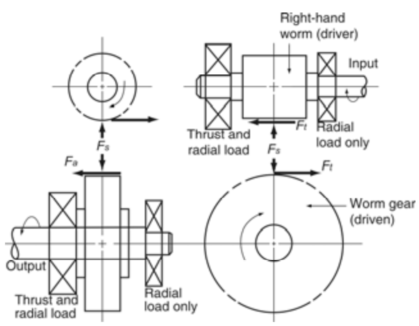
- Driving force on the worm gear, $F_t = \frac{T_o}{r_{wg}}$
- $T_o$ = Output torque
- Separating force, $F_s = \frac{F_t \sin\varphi}{\cos\varphi \cos\lambda - f \sin\lambda}$

where

- $\lambda$ = lead angle
- $\varphi$ = normal pressure angle
- f = coefficient of friction



## Loads from Worm gears






## Loads from Worm gears

- Axial force on the worm gear

$$F_{a(gear)} = F_{t(gear)}\left(\frac{\cos\varphi \sin\lambda + f\cos\lambda}{\cos\varphi \cos\lambda - f \sin\lambda}\right)$$

where

- $\lambda$ = lead angle
- $\varphi$ = normal pressure angle
- f = coefficient of friction



## Loads from Belts and Chains

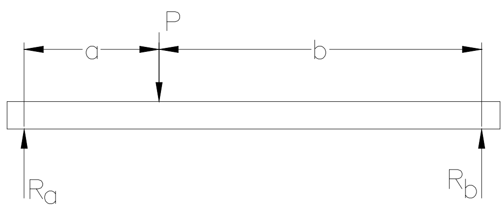
- For a belt, Total load, $F_t = F_f + F_b$
- Net driving force, $F_d = F_f - F_b$
- Driving torque, $T = F_d\, r$
- r = effective radius of pulley or sprocket
- For a chain $F_b = 0$



## Bending of circular shafts

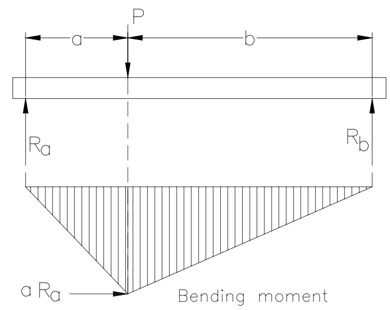
- Shafts transmit power through gears and pulleys
- These produce bending load in addition to torsion
- Use strength of material approach to calculate the reaction forces and bending moments

# Bending of circular shafts






# Bending of circular shafts





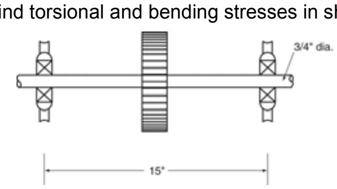

# Shaft Design Problems

- **Step 1:** Calculate the torque on the shaft from power
- **Step 2:** Find the torsional stress in the shaft
- **Step 3:** Calculate the loads coming from gears, belts or chains
- **Step 4:** Calculate the bending moment due to the acting forces. If necessary combine the forces.
- **Step 5:** Calculate the bending stress in the shaft
- **Step 6:** Combine the bending stress and the torsional stress using the theories discussed in chapter 4



**Example Problem 17-1: Design Stresses in Shafts**

- Shaft shown drives a gear set that is transmitting 5 hp at 1750 rpm.
- Shaft is supported in self-aligning ball bearings and gears are both 10 pitch, 40 tooth, 20° spur gears.
- Find torsional and bending stresses in shaft.






**Example Problem 17-1: Design Stresses in Shafts** (cont'd.)

- Find the torsion in the shaft:

(2-6)

$$hp = \frac{Tn}{63{,}000}$$

then:

(17-1)

$$T = \frac{63{,}000\ hp}{n}$$

$$T = \frac{63{,}000\ (5)}{1750}$$

$$T = 180\ in\text{-}lb$$



**Example Problem 17-1: Design Stresses in Shafts** (cont'd.)

- Find the torsional stress in the shaft.
  - First find Z':

(Appendix 3)

$$Z' = \frac{\pi D^3}{16}$$

$$Z' = \frac{\pi (.75\ in)^3}{16}$$

$$Z' = .083\ in^3$$

(3-6)

$$S_s = \frac{T}{Z'}$$

$$S_s = \frac{180\ in\text{-}lb}{.083\ in^3}$$

$$S_s = 2170\ lb/in^2$$

## Example Problem 17-1: Design Stresses in Shafts (cont'd.)

- Find the load at the gear pitch circle:

(11-4)

$$D_p = \frac{N_T}{P_d}$$

$$D_p = \frac{40}{10}$$

$$D_p = 4 \text{ inches}$$

(12-3)

$$F_t = \frac{2T}{D_P}$$

$$F_t = \frac{2\,(180 \text{ in-lb})}{4 \text{ in}}$$

$$F_t = 90 \text{ lb}$$



## Example Problem 17-1: Design Stresses in Shafts (cont'd.)

- Find the resultant force on the shaft:

(12-2)

$$F_r = \frac{F_t}{\cos\theta}$$

$$F_r = \frac{90 \text{ lb}}{\cos 20°}$$

$$F_r = 96 \text{ lb}$$

- Find the maximum moment:

(Appendix 2)

$$M_m = \frac{FL}{4}$$

$$M_m = \frac{96 \text{ lb } (15 \text{ in})}{4}$$

$$M_m = 360 \text{ in-lb}$$



## Example Problem 17-1: Design Stresses in Shafts (cont'd.)

- Find the stress:

$$S = \frac{M}{Z}$$

(Appendix 3)

$$Z = \frac{\pi D^3}{32}$$

$$Z = \frac{\pi (.75 \text{ in})^3}{32}$$

$$Z = .041 \text{in}^3$$

$$S = \frac{M}{Z}$$

$$S = \frac{360 \text{ in-lb}}{.041 \text{ in}^3}$$

$$S = 8780 \text{ lb/in}^2$$



# Combined Stresses in Shafts

- As seen in Chap 4



# Combined maximum shear stress

- $\tau$ = Maximum combined shear stress
- S = normal stress
- $S_S$ = shear stress

$$\tau = \left[ S_S^{\ 2} + \left( \frac{S}{2} \right)^2 \right]^{1/2}$$

- This can be rewritten as

$$\tau = \frac{5.1}{D^3} \left( T^2 + M^2 \right)^{1/2}$$

- T = Torque in the shaft
- M = Maximum moment



## Example Problem 17-2: Combined Stresses in Shafts

• From previous example problem, find the combined stress using the maximum shear stress theorem:

(4-5)

$$\tau = \left( S_s^{\,2} + \left( \frac{S}{2} \right)^2 \right)^{1/2}$$

– Substituting stresses from previous example problem:

$$\tau = \left( (2170 \text{ lb/in}^2)^2 + \left( \frac{8780}{2} \text{ lb/in}^2 \right)^2 \right)^{1/2}$$

$$\tau = 4900 \text{ lb/in}^2$$

– This should be compared to shear stress allowables.

## Maximum Normal Stress Theory

- σ = equivalent combined normal stress
- S = normal stress from bending or axial loads
- $S_S$ = shear or torsional stress

$$\sigma = \frac{S}{2} \pm \left[ S_S^{\,2} + \left( \frac{S}{2} \right)^2 \right]^{1/2}$$

- This can be written as

$$\sigma = \frac{5.1}{D^3}\left[ M + (T^2 + M^2)^{1/2} \right]$$



### Example Problem 17-3: Combined Stresses in Shafts

• From Example Problem 17-1, find the combined stress using the maximum normal stress theory:

$$\sigma = \frac{S}{2} \pm \left( S_s^{\,2} + \left( \frac{S}{2} \right)^2 \right)^{1/2}$$

– Substituting stresses from Example Problem 17-1:

$$\sigma = \frac{8780\, lb/in^2}{2} + \left( (2170\, lb/in^2)^2 + \left( \frac{8780\, in^2}{2} \right)^2 \right)^{1/2}$$

$$\sigma = 9300\, lb/in^2$$

– This should be compared to the normal stress allowable.



## Solid Circular shaft

$$D = \sqrt[3]{\frac{5.1}{\tau}\left( T^2 + M^2 \right)^{1/2}}$$

$$D = \sqrt[3]{\frac{5.1}{\sigma}\left[ M + (T^2 + M^2)^{1/2} \right]}$$



## Critical speeds of shafts

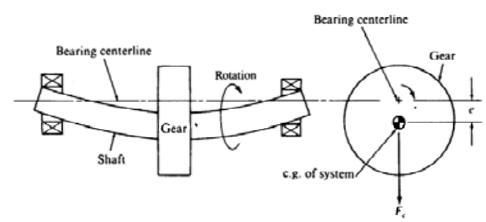
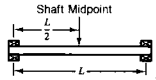
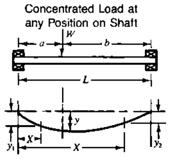



FIGURE 5.9 Eccentricity in a rotating shaft



TABLE 5.3 Bending Moments and Deflections for Simply Supported Shafts

| | Bending Moment at Load | Shaft Deflection |
|---|---|---|
| Concentrated Load at Shaft Midpoint  | $\frac{WL}{4}$ | $y = \frac{WL^3}{48EI}$ (at midspan) |
| Concentrated Load at any Position on Shaft  | $\frac{Wab}{L}$ | $y = \frac{Wa^2b^2}{3EIL}$ (at load)<br>$y_1 = \frac{Wbx}{6EIL}[L^2 - b^2 - x^2]$ (at any distance x to the left of the load)<br>$y_2 = \frac{Wa(L-x)}{6EIL}[L^2 - a^2 - (L-x)^2]$ (at any distance x to the right of the load) |



## Critical speeds of shafts

- Operating speed should be 20% away from the critical speed.
- Vibration frequency, f is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k\,g}{W}}$$

- f = frequency in cycles per second, Hz
- k = force constant, force per inch of deflection
- g = acceleration due to gravity, 386.4 in./$s^2$
- W = weight in pounds, lb

- Change the frequency to rpm
- Critical speed, $N_c = 60 \times f$
- Also k is weight divided by deflection

$$k = \frac{W}{\delta}$$

$$N_c = \frac{60}{2\pi}\sqrt{\frac{W\,g}{W\,\delta}}$$

$$N_c = 187.7\sqrt{\frac{1}{\delta}}$$



# Shaft with n concentrated loads

- Rayleigh's equation is used.

$$N_c = 187.7\sqrt{\frac{W_1\delta_1 + W_2\delta_2 + W_3\delta_3 + \ldots + W_n\delta_n}{W_1\delta_1^2 + W_2\delta_2^2 + W_3\delta_3^2 + \ldots + W_n\delta_n^2}}$$



## Example Problem 17-5: Critical Speed

• Find the estimated critical speed for the shaft in Example Problem 17-1 (assume the entire shaft diameter is ¾ inch).

– First, find deflection:

$$\delta = -\frac{FL^3}{48\,EI}$$ (Appendix 2)

$$I = \frac{\pi D^4}{64}$$ (Appendix 3)

$$I = \frac{\pi(.75\ in)^4}{64}$$

$$I = .016\ in^4$$

$$\delta = -\frac{96\ lb\ (15\ in)^4}{48\ (30 \times 10^6\ lb/in^2)\ (.016\ in^4)}$$

$$\delta = .21\ inch$$



## Example Problem 17-5: Critical Speed (cont'd.)

$$N_c = \frac{188}{\sqrt{\partial}}$$

$$N_c = \frac{188}{\sqrt{.21}}$$ (17-14)

$$N_c = 410\ rpm$$

• This is approximate, and additional multiples would exist at 820, 1230, and 1640 rpm.